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Non-Hermitian matrix Schrödinger equation: Bäcklund–Darboux transformation, Weyl functions and $\mathcal{PT}$ symmetry

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Abstract

The non-Hermitian matrix Schrödinger equation obtained by the Bäcklund–Darboux transformation (BDT) is treated. The potentials, fundamental solutions and Weyl functions are constructed explicitly. A $\mathcal{PT}$ symmetric reduction of the BDT is introduced and this case is studied in greater detail, including potentials, fundamental solutions, bound states, the reality of the discrete spectrum and spontaneous break of the $\mathcal{PT}$ symmetry, the sign-indefinite scalar product, and examples.

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1. Introduction

This paper deals with the matrix Schrödinger equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} y(x, \lambda) - u(x) y(x, \lambda) + \lambda y(x, \lambda) = 0 \qquad (-\infty < x < \infty) \tag{1.1}$$

where $u(x)$ is an $h \times h$ locally summable matrix function, and λ is a spectral parameter. The self-adjoint Schrödinger equation is a classical object of research in physics and mathematics. The growing interest in the non-Hermitian operators is motivated by theoretical and applied reasons. One of the most interesting non-self-adjoint cases is the case of the $\mathcal{PT}$ symmetric potential: $u(x) = u(-x)^*$. The $\mathcal{PT}$ symmetric quantum mechanics of Bender and Bötcher [5] have been actively and variously developed since [5] was published; see [6, 10, 14]. For the first examples of $\mathcal{PT}$ symmetric potentials see, for instance, [7]; interesting recent results and references can be found in [4, 15–17]. The explicit construction of the bound states or energy

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eigenfunctions, i.e. square summable on $\mathbb{R}$ solutions (eigenfunctions) of equation (1.1), is of essential interest in the theory; see [3, 10, 17] and references therein.

Here we study equation (1.1) using the Bäcklund–Darboux transformation (BDT). BDTs are widely used in both spectral and integrable nonlinear equation theories; a large amount of literature and numerous results on BDTs are contained in [1, 8, 11, 21, 22, 33]. Some interesting applications of the Darboux method to the $\mathcal{PT}$ symmetric case can be found in [10, 23, 32] (see also remark 4.5). The version of the BDT that we apply was initially developed in [25, 26]; see further references in [28]. Explicit solutions of the direct and inverse spectral problems and bound states for self-adjoint systems on the semi-axis with the so-called pseudo-exponential potentials have been studied in [12, 13].

We introduce the $\mathcal{PT}$ symmetric reduction of the BDT, we formulate the theorem on the spectrum of the matrix Schrödinger equation obtained via the BDT, and we construct explicitly bound states, fundamental solutions and Weyl functions. The classical approach to the BDT in terms of the operator factorization is closely related to the notion of supersymmetry. In this way, we develop the study of the interplay of supersymmetry and $\mathcal{PT}$ symmetry undertaken in [17]. The scalar non-Hermitian Schrödinger equation is of special interest and the results are new even in the scalar case.

In section 2 we adduce some basic results from [28] on the BDT (GBDT in the terminology of [28]) to make the paper self-contained. Using these results, the expressions for the eigenfunctions and $\mathcal{PT}$ symmetric GBDT reduction are derived. In section 3 we deal with the explicit formulae for the GBDT transformed Schrödinger equation, which can be obtained when the initial equation is trivial. Theorem 3.7 gives sufficient conditions, when λ does not belong to the discrete spectrum of equation (1.1). Finally, section 4 is dedicated to the explicit formulae for the case of the $\mathcal{PT}$ symmetric u: potentials, fundamental solutions, bound states, the reality of the spectrum and the spontaneous break of the $\mathcal{PT}$ symmetry, the sign-indefinite scalar product, and examples. Section 5 contains the conclusion.

We denote by $\mathbb{C}$ the complex plane and by $\mathbb{R}$ the real axis.

2. BDT and $\mathcal{PT}$ symmetric reduction

We consider an $m \times m$ first-order system

$$w'(x,\lambda) = G(x,\lambda)w(x,\lambda) \qquad G(x,\lambda) = -\sum_{k=0}^{r} \lambda^k q_k(x) \tag{2.1}$$

where $w' = \frac{d}{dx}w$, and the coefficients $q_k(x)$ are $m \times m$ locally summable matrix functions. For simplicity we suppose that G is polynomial in λ and λ does not depend on x, t (the case of G rationally depending on $\lambda(x,t)$ was treated in [27]). The function w in equation (2.1) is an absolutely continuous matrix function; it may be either a fundamental solution or vector function, in particular. (We normalize the $m \times m$ fundamental solution of equation (2.1) by the initial condition $w(0,\lambda) = I_m$, where I_m is the $m \times m$ identity matrix.) Our version of the BDT (GBDT) is determined by the choice of the five parameter matrices: three square matrices A_1, A_2, and $S(0)$ ($\det S(0) \neq 0$) of order n, and two $n \times m$ matrices $\Pi_1(0)$ and $\Pi_2(0)$. These parameter matrices should satisfy the operator identity

$$A_1 S(0) - S(0) A_2 = \Pi_1(0)\Pi_2(0)^* \tag{2.2}$$

where R^* denotes the conjugate transpose for some matrix R (conjugate for scalar). Suppose that such parameter matrices are fixed. Then we can introduce matrix functions $\Pi_1(x)$, $\Pi_2(x)$

and $S(x)$ with the values $\Pi_1(0)$, $\Pi_2(0)$ and $S(0)$ at $x = 0$ as the solutions of the linear differential equations:

$$\Pi_1'(x) = \sum_{p=0}^{r} A_1^p \Pi_1(x) q_p(x) \qquad \Pi_2'(x) = -\sum_{p=0}^{r} (A_2^*)^p \Pi_2(x) q_p(x)^*$$
$$S'(x) = \sum_{p=1}^{r} \sum_{j=1}^{p} A_1^{p-j} \Pi_1(x) q_p(x) \Pi_2(x)^* A_2^{j-1}. \tag{2.3}$$

Notice that equations (2.3) are constructed in such a way that the identity

$$A_1 S(x) - S(x) A_2 = \Pi_1(x) \Pi_2(x)^* \tag{2.4}$$

follows from equations (2.2) and (2.3) for all x in the containing zero connected domain, where the coefficients q_k are defined. (The relation is obtained by the direct differentiation of both sides of equation (2.4).) Moreover, using equation (2.3) Π_2^* is a 'generalized eigenfunction' of system (2.1) corresponding to the generalized (matrix) eigenvalue A_2, and Π_1 is a 'generalized eigenfunction' of a dual system, corresponding to the matrix eigenvalue A_1. Assuming that $\det S(x) \not\equiv 0$ we can define a matrix function

$$w_A(x, \lambda) = I_m - \Pi_2(x)^* S(x)^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1(x) \tag{2.5}$$

where $\lambda \notin \sigma(A_1)$ (σ denotes spectrum).

Theorem 2.1 [26, 28]**.** *Suppose that matrix functions w, Π_1, Π_2 and S satisfy equations (2.1)–(2.3). Then, in the points of invertibility of S, the matrix function w_A satisfies the system*

$$w_A'(x, \lambda) = \widetilde{G}(x, \lambda) w_A(x, \lambda) - w_A(x, \lambda) G(x, \lambda) \tag{2.6}$$

where $\widetilde{G}(x, \lambda) = -\sum_{k=0}^{r} \lambda^k \widetilde{q}_k(x)$, and the coefficients $\widetilde{q}_k$ are given by the formulae

$$\widetilde{q}_k(x) = q_k(x) - \sum_{p=k+1}^{r} (q_p(x) Y_{p-k-1}(x) - X_{p-k-1}(x) q_p(x)$$
$$+ \sum_{j=k+2}^{p} X_{p-j}(x) q_p(x) Y_{j-k-2}(x)) \tag{2.7}$$

$$X_k(x) = \Pi_2(x)^* S(x)^{-1} A_1^k \Pi_1(x) \qquad Y_k(x) = \Pi_2(x)^* A_2^k S(x)^{-1} \Pi_1(x). \tag{2.8}$$

According to theorem 2.1 the multiplication by w_A transforms the fundamental solution w of equation (2.1) into the fundamental solution $\widetilde{w} = w_A w$ of the system $\widetilde{w}' = \widetilde{G}\widetilde{w}$ with the coefficients $\widetilde{q}_k$ of $\widetilde{G}$ given by equation (2.7). This transformation of the fundamental solution w and coefficients q_k is called the GBDT. The matrix function w_A is the so-called Darboux matrix. The representation of the Darboux matrix in the form (2.5) proved useful in the spectral and bispectral theories. Transfer matrix functions of the form $w_A = I - C(A - \lambda I_n)^{-1} B$ are a well-known tool in system theory. Matrix functions of the form (2.5) with the additional property (2.4) were introduced by Sakhnovich [30, 31] in the context of his method of operator identities. If $S = I$ these coincide with the well-known characteristic matrix functions [18].

Under the conditions of theorem 2.1 we also have [28]

$$(\Pi_2^* S^{-1})'(x) = -\sum_{p=0}^{r} \widetilde{q}_p(x) \Pi_2(x)^* S(x)^{-1} A_1^p \tag{2.9}$$

$$(S^{-1}\Pi_1)'(x) = \sum_{p=0}^{r} A_2^p S(x)^{-1} \Pi_1(x) \widetilde{q}_p(x). \tag{2.10}$$

According to equation (2.9) multiplication by S^{-1} transforms the eigenfunction Π_2^* corresponding to the generalized eigenvalue A_2 of the initial system into the eigenfunction $\Pi_2^* S^{-1}$ corresponding to the generalized eigenvalue A_1 of the transformed system. According to equation (2.10) multiplication by S^{-1} also transforms the eigenfunction Π_1 corresponding to the generalized eigenvalue A_1 of the dual system into the eigenfunction $S^{-1}\Pi_1$ corresponding to the generalized eigenvalue A_2 of the transformed dual system. We can see that the operator of multiplication by S^{-1} is some kind of instanton generator in the GBDT.

We now consider system (2.1) and put $m = 2h$, $r = 1$,

$$q_1 = \begin{bmatrix} 0 & 0 \\ I_h & 0 \end{bmatrix} \qquad q_0(x) = -\begin{bmatrix} 0 & I_h \\ u(x) & 0 \end{bmatrix}. \tag{2.11}$$

Solution w of system (2.1), with the coefficients given by equation (2.11), can be written down in block form: $w = \left[\begin{smallmatrix} y \\ \hat{y} \end{smallmatrix}\right]$. Hence we rewrite (2.1) as $y'(x,\lambda) = \hat{y}(x,\lambda)$, $\hat{y}'(x,\lambda) = -\lambda y(x,\lambda) + u(x)y(x,\lambda)$, i.e. equation (1.1) is fulfilled. So system (2.1) and (2.11) is equivalent to the Schrödinger equation (1.1). The following proposition is a corollary of theorem 2.1.

Proposition 2.2 [28]**.** *Let a matrix function $y(x,\lambda)$ satisfy the Schrödinger equation (1.1) and put*

$$\tilde{y}(x,\lambda) = [I_h \quad 0]\tilde{w}(x,\lambda) \tag{2.12}$$

where $\tilde{w}$ is the GBDT of the solution $w = \left[\begin{smallmatrix} y \\ y' \end{smallmatrix}\right]$ of system (2.1) and (2.11). Then $\tilde{y}$ satisfies the Schrödinger equation

$$\tilde{y}''(x,\lambda) - \tilde{u}(x)\tilde{y}(x,\lambda) + \lambda\tilde{y}(x,\lambda) = 0 \tag{2.13}$$

where

$$\tilde{u}(x) = u(x) - 2X_{012}'(x) \qquad X_{012}'(x) = X_{022}(x) - X_{011}(x) - X_{012}(x)^2 \tag{2.14}$$

and X_{0kj} are the $h \times h$ blocks of the matrix $X_0 = \Pi_2^* S^{-1}\Pi_1$.

Instead of the Schrödinger equation (2.13) we can talk about the Schrödinger operator $\widetilde{L} = -\frac{\mathrm{d}}{\mathrm{d}x^2} + \widetilde{u}$ with a properly defined domain.

Now we present Π_1 and Π_2 in the block form $\Pi_1 = [\Phi_1 \quad \Phi_2]$ and $\Pi_2 = [\Psi_1 \quad \Psi_2]$, where Φ_k, Ψ_k ($k = 1, 2$) are the $n \times h$ matrix functions. We shall need an auxiliary proposition that follows from equations (2.9) and (2.10).

Proposition 2.3. *Let matrix functions Π_1, Π_2, S and $\widetilde{u}$ be defined by the parameter matrices and by system (2.1) and (2.11) as in proposition 2.2. Then we have*

$$(\Psi_1(x)^* S(x)^{-1})'' = -\Psi_1(x)^* S(x)^{-1} A_1 + \widetilde{u}(x)\Psi_1(x)^* S(x)^{-1} \tag{2.15}$$

$$(S(x)^{-1}\Phi_2(x))'' = -A_2 S(x)^{-1}\Phi_2(x) + S(x)^{-1}\Phi_2(x)\widetilde{u}(x). \tag{2.16}$$

Proof. For the sake of brevity we sometimes omit the argument x in our calculations. From equation (2.7) it follows that

$$\widetilde{q}_1 = q_1 \qquad \widetilde{q}_0 = q_0 + X_0 q_1 - q_1 X_0 = \begin{bmatrix} X_{012} & -I_h \\ -u + X_{022} - X_{011} & -X_{012} \end{bmatrix}. \tag{2.17}$$

Now we rewrite equation (2.9) as

$$(\Psi_1^* S^{-1})' = -X_{012}\Psi_1^* S^{-1} + \Psi_2^* S^{-1} \tag{2.18}$$

$$(\Psi_2^* S^{-1})' = -\Psi_1^* S^{-1} A_1 + (u + X_{011} - X_{022})\Psi_1^* S^{-1} + X_{012}\Psi_2^* S^{-1}. \tag{2.19}$$

Therefore, by differentiating the left-hand side in equation (2.18) we obtain

$$\begin{aligned}(\Psi_1^*S^{-1})'' &= -X_{012}'\Psi_1^*S^{-1} - X_{012}(-X_{012}\Psi_1^*S^{-1} + \Psi_2^*S^{-1}) - \Psi_1^*S^{-1}A_1 \\ &\quad + (u + X_{011} - X_{022})\Psi_1^*S^{-1} + X_{012}\Psi_2^*S^{-1} \\ &= -\Psi_1^*S^{-1}A_1 + \big(u + X_{011} - X_{022} + X_{012}^2 - X_{012}'\big)\Psi_1^*S^{-1}. \end{aligned} \tag{2.20}$$

In view of equation (2.14), equation (2.20) yields equation (2.15). Quite analogously from equation (2.10) we obtain

$$\begin{aligned}(S^{-1}\Phi_2)' &= -S^{-1}\Phi_1 - S^{-1}\Phi_2X_{012} \\ (S^{-1}\Phi_1)' &= A_2S^{-1}\Phi_2 + S^{-1}\Phi_1X_{012} + S^{-1}\Phi_2(-u + X_{022} - X_{011})\end{aligned} \tag{2.21}$$

and equation (2.16) follows from equation (2.21). □

If u is $\mathcal{PT}$ symmetric, i.e. $u(x) = u(-x)^*$, then the coefficients q_p have the property:

$$q_p(x) = Jq_p(-x)^*J \quad (p = 0, 1) \qquad J = J^* = J^{-1} = \begin{bmatrix} 0 & I_h \\ I_h & 0 \end{bmatrix}. \tag{2.22}$$

Suppose that the parameter matrices satisfy additional restrictions:

$$A_1 = A_2^* =: A \qquad S(0) = S(0)^* \qquad \Pi_1(0) = \mathrm{i}\Pi_2(0)J =: \Pi(0). \tag{2.23}$$

Then the operator identity (2.4) takes the form

$$AS(0) - S(0)A^* = \mathrm{i}\Pi(0)J\Pi(0)^*. \tag{2.24}$$

According to equations (2.3), (2.22) and (2.23) we have

$$(\mathrm{i}\Pi_2(-x)J)' = \mathrm{i}\sum_{p=0}^{1} A_1^p\Pi_2(-x)JJq_p(-x)^*J = \mathrm{i}\sum_{p=0}^{1} A_1^p\Pi_2(-x)Jq_p(x).$$

Therefore, taking into account $\Pi_1(0) = \mathrm{i}\Pi_2(0)J$, we obtain

$$\Pi_1(x) = \mathrm{i}\Pi_2(-x)J \qquad \Phi_1(x) = \mathrm{i}\Psi_2(-x) \qquad \Phi_2(x) = \mathrm{i}\Psi_1(-x). \tag{2.25}$$

Quite analogously we derive

$$S(x) = S(-x)^*. \tag{2.26}$$

From the first relation in equation (2.14) and equations (2.25) and (2.26), it follows that

$$X_{012}(x) = -X_{012}(-x)^* \qquad \widetilde{u}(x) = \widetilde{u}(-x)^*. \tag{2.27}$$

Proposition 2.4.

(i) Let equalities (2.23) hold and let the parameter matrices $A, \Pi(0)$ and $S(0)$ satisfy identity (2.24). Suppose that $u(x) = u(-x)^*$. Then equalities (2.27) are valid, i.e. $\widetilde{u}$ given by equalities (2.14) (the GBDT of the potential u) is $\mathcal{PT}$ symmetric also.

(ii) Suppose additionally that $Af = af$ ($f \in \mathbb{C}^n$) and $\det S(x) \neq 0$ ($-\infty < x < \infty$). Then the vector functions $\Psi_1(x)^*S(x)^{-1}f$ and $(S(-x)^{-1}\Phi_2(-x))^*f$ are eigenfunctions of equation (2.13) (of the operator $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \widetilde{u}(x)$) corresponding to the eigenvalue a.

Proof. The statement (i) was proved above and the statement (ii) about $\Psi_1^*S^{-1}f$ is immediate from proposition 2.3. Using equations (2.23) and (2.27) we rewrite equation (2.16) as

$$((S(-x)^{-1}\Phi_2(-x))^*)'' = -(S(-x)^{-1}\Phi_2(-x))^*A_1 + \widetilde{u}(x)(S(-x)^{-1}\Phi_2(-x))^*. \tag{2.28}$$

Now we can see that $(S(-x)^{-1}\Phi_2(-x))^*f$ is an eigenfunction also. □

The problem of the appropriate for the $\mathcal{PT}$ symmetric model scalar product is actively discussed in the literature. A space $\mathcal{F}$ of the state vectors for the scalar Schrödinger equation with a sign-indefinite scalar product and corresponding transition probability amplitude has been introduced in the interesting paper [16]. Quite analogously we put

$$(\psi, \widetilde{\psi}) = \int_{-\infty}^{\infty} \widetilde{\psi}(-x)^* \psi(x)\, \mathrm{d}x. \tag{2.29}$$

If ψ and $\widetilde{\psi}$ are bound states with different eigenvalues, then this scalar product turns to zero.

Remark 2.5. If the conditions of (i) of proposition 2.4 hold, then the scalar product $(y(x, \lambda_1) f_1, y(x, \lambda_2) f_2)$ is given by

$$(y(x, \lambda_1) f_1, y(x, \lambda_2) f_2) = (\lambda_2^* - \lambda_1)^{-1} \Big(\lim_{b \to \infty} f_2^* (\widetilde{w}(-b, \lambda_2)^* J \widetilde{w}(b, \lambda_1)) f_1 - \lim_{a \to -\infty} f_2^* (\widetilde{w}(-a, \lambda_2)^* J \widetilde{w}(a, \lambda_1)) f_1 \Big) \tag{2.30}$$

where $\widetilde{w} = w_A w$, and y is defined by equality (2.12). Indeed, according to equalities (2.25) and (2.26) we have $X_{011}(-x)^* = -X_{022}(x)$. Hence, in view of equalities (2.27), equalities (2.17) yields

$$J\widetilde{q}_0(x) - \widetilde{q}_0(-x)^* J = 0 \qquad \lambda_1 J q_1 - \lambda_2^* q_1^* J = (\lambda_1 - \lambda_2^*) \begin{bmatrix} I_h & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.31}$$

From equation (2.31) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} (\widetilde{w}(-x, \lambda_2)^* J \widetilde{w}(x, \lambda_1)) = (\lambda_2^* - \lambda_1) \widetilde{w}(-x, \lambda_2)^* \begin{bmatrix} I_h & 0 \\ 0 & 0 \end{bmatrix} \widetilde{w}(x, \lambda_1). \tag{2.32}$$

Finally, from equations (2.12), (2.29) and (2.32) we obtain (2.30).

It would be interesting to use [29] to obtain some analogues of the above results in the case of several variables.

3. Explicit formulae

In this section we consider the case of the trivial initial Schrödinger equation: $u = 0$. We can see easily that the matrix function

$$y(x, \lambda) = [I_h \quad 0] T(\mu)\, \mathrm{e}^{\mathrm{i}\mu x j} \qquad (\lambda = \mu^2) \tag{3.1}$$

where

$$T(\mu) = \begin{bmatrix} I_h & I_h \\ \mathrm{i}\mu I_h & -\mathrm{i}\mu I_h \end{bmatrix} \qquad j = \begin{bmatrix} I_h & 0 \\ 0 & -I_h \end{bmatrix} \tag{3.2}$$

satisfies the Schrödinger equation $y''(x, \lambda) + \lambda y(x, \lambda) = 0$ with $u = 0$. Moreover, we have $y'(x, \lambda) = [0 \quad I_h] T(\mu)\, \mathrm{e}^{\mathrm{i}\mu x j}$, i.e.,

$$\begin{bmatrix} y(x, \lambda) \\ y'(x, \lambda) \end{bmatrix} = T(\mu)\, \mathrm{e}^{\mathrm{i}\mu x j}. \tag{3.3}$$

Now we assume that

$$A_1 = \omega_1^2 \qquad A_2^* = \omega_2^2. \tag{3.4}$$

In this case, the GBDT of the solution and potential is constructed explicitly up to matrix exponents. The first two equations in (2.3) can be rewritten as $\Phi_1' = \omega_1^2 \Phi_2$, $\Phi_2' = -\Phi_1$, $\Psi_1' = \Psi_2$, and $\Psi_2' = -\omega_2^2 \Psi_1$. We introduce $2n \times 2n$ matrices

$$\Omega_p = \mathrm{i} \begin{bmatrix} \omega_p & 0 \\ 0 & -\omega_p \end{bmatrix} \qquad T(\omega_p) = \begin{bmatrix} I_n & I_n \\ \mathrm{i}\omega_p & -\mathrm{i}\omega_p \end{bmatrix} \quad (p = 1, 2). \tag{3.5}$$

We see that the $n \times 2h$ matrix functions $\Pi_1 = [\Phi_1 \quad \Phi_2]$ and $\Pi_2 = [\Psi_1 \quad \Psi_2]$ given by the equalities

$$\begin{bmatrix}\Phi_2(x)\\ \Phi_1(x)\end{bmatrix} = T(\omega_1)\,\mathrm{e}^{-x\Omega_1}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} \qquad \begin{bmatrix}\Psi_1(x)\\ \Psi_2(x)\end{bmatrix} = T(\omega_2)\,\mathrm{e}^{x\Omega_2}\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix} \tag{3.6}$$

where θ_p and ν_p are $n \times h$ constant matrices, satisfy equations (2.3). For q_0 and q_1 defined by equalities (2.11), the third equation in (2.3) takes the form

$$S'(x) = \Phi_2(x)\Psi_1(x)^*. \tag{3.7}$$

From proposition 2.2 follows:

Corollary 3.1. *Let matrix functions Π_1, Π_2 and S be given by equations (3.6) and (3.7), and let equation (2.2) hold. Then the matrix function*

$$\widetilde{y}(x,\lambda) = [I_h \quad 0]w_A(x,\lambda)T(\mu)\,\mathrm{e}^{\mathrm{i}\mu x j} \tag{3.8}$$

satisfies the matrix Schrödinger equation (2.13), where

$$\widetilde{u}(x) = -2X_{012}'(x) \qquad X_0(x) = \Pi_2(x)^* S(x)^{-1}\Pi_1(x). \tag{3.9}$$

Proof. In view of equality (3.3), equality (2.12) now takes the form (3.8), and the statement of the corollary is immediate. □

Definition 3.2. *An $h \times 2h$ solution $v(x,\lambda)$ of the matrix Schrödinger equation is called the fundamental solution if the initial conditions*

$$v(0,\lambda) = [I_h \quad 0] \qquad v'(0,\lambda) = [0 \quad I_h] \tag{3.10}$$

are valid.

To construct the fundamental solution, we notice that

$$T(\mu)^{-1} = \frac{\mathrm{i}}{2\mu}\begin{bmatrix}-\mathrm{i}\mu I_h & -I_h\\ -\mathrm{i}\mu I_h & I_h\end{bmatrix} \qquad \mathrm{i}\mu T(\mu) j T(\mu)^{-1} = \begin{bmatrix}0 & I_h\\ -\lambda I_h & 0\end{bmatrix} = G(\lambda) \tag{3.11}$$

where q_p that define G are given by equalities (2.11), $u = 0$. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}T(\mu)\,\mathrm{e}^{\mathrm{i}\mu x j} = G(\lambda)T(\mu)\,\mathrm{e}^{\mathrm{i}\mu x j}. \tag{3.12}$$

In view of equations (2.6), (2.17) and (3.12) for $\widetilde{y}$ given by equation (3.8), we obtain

$$\begin{aligned}\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}x} &= -[I_h \quad 0](\lambda\widetilde{q}_1 + \widetilde{q}_0(x))w_A(x,\lambda)T(\mu)\,\mathrm{e}^{\mathrm{i}\mu x j}\\ &= [-X_{012}(x) \quad I_h]w_A(x,\lambda)T(\mu)\,\mathrm{e}^{\mathrm{i}\mu x j}.\end{aligned} \tag{3.13}$$

According to equations (3.8) and (3.13), the matrix function

$$v(x,\lambda) = \widetilde{y}(x,\lambda)T(\mu)^{-1}w_A(0,\lambda)^{-1}T_0 \qquad T_0 = \begin{bmatrix}I_h & 0\\ X_{012}(0) & I_h\end{bmatrix} \tag{3.14}$$

satisfies equation (3.10), i.e. v is a fundamental solution. From equalities (2.4) and (2.5), it follows [30] that

$$w_A(x,\lambda)^{-1} = I_m + \Pi_2(x)^*(A_2 - \lambda I_n)^{-1}S(x)^{-1}\Pi_1(x). \tag{3.15}$$

Thus the right-hand side of equation (3.14) is defined for $\lambda \in \mathbb{C}\backslash(\sigma(A_1) \cup \sigma(A_2) \cup \{0\})$.

Similar to the scalar self-adjoint case, Weyl–Titchmarsh functions can be introduced in the matrix and non-self-adjoint cases; see, for instance, [9, 12, 24, 31] and references therein.

Definition 3.3. *An $h \times h$ matrix function $\varphi(\lambda)$ is called a Weyl function of the Schrödinger equation (1.1) on $(0, \infty)$ and we write $\varphi \in \mathcal{W}_+$ if the entries of $v(x, \lambda)\begin{bmatrix} I_h \\ \varphi(\lambda) \end{bmatrix}$ belong to $L^2(0, \infty)$. If these entries belong to $L^2(-\infty, 0)$ we write $\varphi \in \mathcal{W}_-$, i.e. φ is a Weyl function of equation (1.1) on $(-\infty, 0)$.*

We introduce a $2h \times 2h$ matrix function Υ with the $h \times h$ blocks Υ_{kj} by the equality

$$\Upsilon(\lambda) = \{\Upsilon_{kj}\}_{k,j=1}^2 = T_0^{-1} w_A(0, \lambda) T(\mu). \tag{3.16}$$

Furthermore, we fix the branch of the square root $\mu = \sqrt{\lambda}$ and assume that

$$\mu \in \mathbb{C}_+ \qquad \text{if} \quad \lambda \notin [0, \infty) \qquad \mu \in [0, \infty) \qquad \text{if} \quad \lambda \in [0, \infty) \tag{3.17}$$

where $\mathbb{C}_+$ is the open upper half-plane. According to equalities (3.14) and (3.16) we have

$$\begin{aligned} v(x, \lambda)\begin{bmatrix} \Upsilon_{11}(\lambda) \\ \Upsilon_{21}(\lambda) \end{bmatrix} &= \mathrm{e}^{\mathrm{i}\mu x}[I_h \quad 0] w_A(x, \lambda)\begin{bmatrix} I_h \\ \mathrm{i}\mu I_h \end{bmatrix} \\ v(x, \lambda)\begin{bmatrix} \Upsilon_{12}(\lambda) \\ \Upsilon_{22}(\lambda) \end{bmatrix} &= \mathrm{e}^{-\mathrm{i}\mu x}[I_h \quad 0] w_A(x, \lambda)\begin{bmatrix} I_h \\ -\mathrm{i}\mu I_h \end{bmatrix}. \end{aligned} \tag{3.18}$$

Using relations (3.17) and (3.18), we can find various sufficient conditions when

$$\Upsilon_{21}(\lambda)\Upsilon_{11}(\lambda)^{-1} \in \mathcal{W}_+ \qquad \Upsilon_{22}(\lambda)\Upsilon_{12}(\lambda)^{-1} \in \mathcal{W}_-.$$

Theorem 3.4. *Let matrix functions Π_1, Π_2, S and $\widetilde{u}$ be given by equations (3.6), (3.7) and (3.9). Let equalities (2.2) and (3.4) hold, and suppose additionally that $\sigma(\omega_p) \in \mathbb{R}$ $(p = 1, 2)$, $\det S(x) \neq 0$, and for each $\varepsilon > 0$ the relation*

$$\|S(x)^{-1}\| < C_\varepsilon \, \mathrm{e}^{\varepsilon|x|} \qquad (-\infty < x < \infty) \tag{3.19}$$

is valid. Then, for the Schrödinger equation (2.13) we have $\Upsilon_{21}(\lambda)\Upsilon_{11}(\lambda)^{-1} \in \mathcal{W}_+$, $\Upsilon_{22}(\lambda)\Upsilon_{12}(\lambda)^{-1} \in \mathcal{W}_-$, where the Weyl functions are considered on $\mathbb{C}\backslash\Gamma$ and Γ consists of $[0, \infty)$ and zeros of $\det \Upsilon_{11}(\lambda)$, $\det \Upsilon_{12}(\lambda)$.

Proof. From equation (3.6) it follows that if $\sigma(\omega_p) \in \mathbb{R}$ then the entries of Π_1 and Π_2 can be presented in the form $\sum_{k=1}^N \mathrm{e}^{\mathrm{i}\alpha_k x} P_k(x)$, where $\alpha_k \in \mathbb{R}$ and P_k are polynomials. Therefore, using relations (2.5) and (3.19) for any $\mu \in \mathbb{C}_+$ the matrix functions $\mathrm{e}^{\mathrm{i}\mu x} w_A(x, \lambda)$ and $\mathrm{e}^{-\mathrm{i}\mu x} w_A(x, \lambda)$ decay exponentially when x tends to $+\infty$ and $-\infty$, respectively. Now, using equation (3.18) the statement of the theorem is immediate. □

Notice that, in view of equation (3.6) we have

$$\begin{aligned} \Phi_1(0) &= \mathrm{i}\omega_1(\theta_1 - \theta_2) \qquad & \Phi_2(0) &= \theta_1 + \theta_2 \\ \Psi_1(0) &= \nu_1 + \nu_2 \qquad & \Psi_2(0) &= \mathrm{i}\omega_2(\nu_1 - \nu_2). \end{aligned} \tag{3.20}$$

Remark 3.5. According to equation (3.6) if a $2n \times 2n$ matrix s satisfies the operator identity

$$s\Omega_2^* - \Omega_1 s = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} [\nu_1^* \quad \nu_2^*] \tag{3.21}$$

then the matrix function $S(x)$ of the form

$$S(x) = P\,\mathrm{e}^{-x\Omega_1} s\,\mathrm{e}^{x\Omega_2^*} P^* \qquad P = [I_n \quad I_n] \tag{3.22}$$

satisfies equation (3.7). Moreover, for $\widetilde{P} = [I_n \ \ -I_n]$ from identity (3.21) it follows that

$$\widetilde{P}(s\Omega_2^* - \Omega_1 s)P^* = -\mathrm{i}(\widetilde{P}s\widetilde{P}^*\omega_2^* + \omega_1 P s P^*) = (\theta_1 - \theta_2)(\nu_1 + \nu_2)^* \tag{3.23}$$

$$P(s\Omega_2^* - \Omega_1 s)\widetilde{P}^* = -\mathrm{i}(P s P^*\omega_2^* + \omega_1 \widetilde{P}s\widetilde{P}^*) = (\theta_1 + \theta_2)(\nu_1 - \nu_2)^*. \tag{3.24}$$

Finally, using equalities (3.20), (3.23) and (3.24), we derive

$$\begin{aligned}\omega_1^2 PsP^* - PsP^*(\omega_2^*)^2 &= \omega_1(\omega_1 PsP^* + \widetilde{P}s\widetilde{P}^*\omega_2^*) - (\omega_1\widetilde{P}s\widetilde{P}^* + PsP^*\omega_2^*)\omega_2^* \\ &= \mathrm{i}(\omega_1(\theta_1-\theta_2)(\nu_1+\nu_2)^* - (\theta_1+\theta_2)(\nu_1-\nu_2)^*\omega_2^*) \\ &= \Pi_1(0)\Pi_2(0)^*. \end{aligned} \tag{3.25}$$

Thus, the matrix function $S(x)$ given by equation (3.22) satisfies equation (2.2) also.

Equality (3.22) defines S more explicitly than equations (2.2) and (3.7), and conditions of invertibility of S on $\mathbb{R}$ can be formulated now in terms of s. In view of equations (3.6) and (3.22) we have

$$\begin{aligned}&\Phi_2(x) = \mathrm{e}^{-\mathrm{i}x\omega_1}\theta_1 + \mathrm{e}^{\mathrm{i}x\omega_1}\theta_2 \qquad \Psi_1(x) = \mathrm{e}^{\mathrm{i}x\omega_2}\nu_1 + \mathrm{e}^{-\mathrm{i}x\omega_2}\nu_2 \\ &S(x) = [\mathrm{e}^{-\mathrm{i}x\omega_1} \quad \mathrm{e}^{\mathrm{i}x\omega_1}]s\begin{bmatrix}\mathrm{e}^{-\mathrm{i}x\omega_2^*}\\ \mathrm{e}^{\mathrm{i}x\omega_2^*}\end{bmatrix}.\end{aligned} \tag{3.26}$$

Consider a simple example.

Example 3.6. For $n = 1$, equation (3.21) takes the form

$$-\mathrm{i}(\omega_2^*\,\mathrm{diag}\{1,-1\}s + \omega_1 s\,\mathrm{diag}\{1,-1\}) = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\begin{bmatrix}\nu_1^* & \nu_2^*\end{bmatrix} \tag{3.27}$$

where diag denotes diagonal matrix, and θ_p and ν_p are row vectors from $\mathbb{C}^h$. If $\omega_1 \neq \pm\omega_2^*$, then from equation (3.27) we obtain

$$s = \mathrm{i}\begin{bmatrix}\dfrac{\theta_1\nu_1^*}{\omega_1+\omega_2^*} & \dfrac{\theta_1\nu_2^*}{\omega_2^*-\omega_1}\\ \dfrac{\theta_2\nu_1^*}{\omega_1-\omega_2^*} & -\dfrac{\theta_2\nu_2^*}{\omega_1+\omega_2^*}\end{bmatrix}. \tag{3.28}$$

Let us simplify the conditions further and assume that

$$\omega_1 = \omega_2 = \omega \in \mathbb{R}\backslash 0 \qquad \theta_2\nu_1^* = \theta_1\nu_2^* = 0 \qquad (\theta_p, \nu_p \in \mathbb{C}^h) \qquad |\theta_1\nu_1^*| \neq |\theta_2\nu_2^*|. \tag{3.29}$$

Then we can put $s = \mathrm{diag}\frac{\mathrm{i}}{2\omega}\{\theta_1\nu_1^*, -\theta_2\nu_2^*\}$. Hence, using the last relations in formulae (3.26) and (3.29) we have

$$S(x) = \frac{\mathrm{i}}{2\omega}(\mathrm{e}^{-2\mathrm{i}x\omega}\theta_1\nu_1^* - \mathrm{e}^{2\mathrm{i}x\omega}\theta_2\nu_2^*) \qquad \det S \neq 0 \tag{3.30}$$

and thus inequality (3.19) holds. Therefore, the conditions of theorem 3.4 are fulfilled. It easily follows from equations (3.9), (3.26) and (3.30) that

$$\begin{aligned}\widetilde{u}(x) = -8\omega^2(&\mathrm{e}^{-2\mathrm{i}x\omega}\theta_1\nu_1^* - \mathrm{e}^{2\mathrm{i}x\omega}\theta_2\nu_2^*)^{-2}(\mathrm{e}^{-2\mathrm{i}x\omega}\theta_1\nu_1^* + \mathrm{e}^{2\mathrm{i}x\omega}\theta_2\nu_2^*)(\nu_1^*\theta_2 + \nu_2^*\theta_1) \\ &+ 2(\theta_1\nu_1^*\nu_2^*\theta_2 + \theta_2\nu_2^*\nu_1^*\theta_1)).\end{aligned} \tag{3.31}$$

To write down Weyl functions of equations (2.13) and (3.31), we notice that $S(0)^{-1} = 2\mathrm{i}\omega(\theta_2\nu_2^* - \theta_1\nu_1^*)^{-1}$ and put $Z_1 = 2\mathrm{i}\omega(\theta_2\nu_2^* - \theta_1\nu_1^*)^{-1}(\nu_1+\nu_2)^*$,

$$Z_2 = 2\mathrm{i}\omega(\theta_2\nu_2^* - \theta_1\nu_1^*)^{-1}(Z_1(\theta_1+\theta_2)(\nu_1+\nu_2)^* + \mathrm{i}\omega(\nu_1-\nu_2)^*).$$

Then Υ_{kj} defined in equality (3.16) are given by the formulae

$$\begin{aligned}&\Upsilon_{21}(\lambda) = \mathrm{i}\mu I_h - Z_1(\theta_1+\theta_2) + (\omega^2-\lambda)^{-1}Z_2(\mathrm{i}\mu(\theta_1+\theta_2) + \mathrm{i}\omega(\theta_1-\theta_2)) \\ &\Upsilon_{11}(\lambda) = I_h - (\omega^2-\lambda)^{-1}Z_1(\mathrm{i}\mu(\theta_1+\theta_2) + \mathrm{i}\omega(\theta_1-\theta_2)) \\ &\Upsilon_{22}(\lambda) = -\mathrm{i}\mu I_h - Z_1(\theta_1+\theta_2) - (\omega^2-\lambda)^{-1}Z_2(\mathrm{i}\mu(\theta_1+\theta_2) - \mathrm{i}\omega(\theta_1-\theta_2)) \\ &\Upsilon_{12}(\lambda) = I_h + (\omega^2-\lambda)^{-1}Z_1(\mathrm{i}\mu(\theta_1+\theta_2) - \mathrm{i}\omega(\theta_1-\theta_2)).\end{aligned}$$

The expressions for the Weyl functions $\Upsilon_{21}(\lambda)\Upsilon_{11}(\lambda)^{-1} \in \mathcal{W}_+$ and $\Upsilon_{22}(\lambda)\Upsilon_{12}(\lambda)^{-1} \in \mathcal{W}_-$ are now immediate.

In the generic case it is necessary that

$$f \in \mathrm{Im}\begin{bmatrix}\Upsilon_{11}(\lambda)\\ \Upsilon_{21}(\lambda)\end{bmatrix} \quad \text{for} \quad v(x,\lambda)f \in L_h^2(0,\infty) \quad \text{and} \quad f \in \mathrm{Im}\begin{bmatrix}\Upsilon_{12}(\lambda)\\ \Upsilon_{22}(\lambda)\end{bmatrix}$$
$$\text{for} \quad v(x,\lambda)f \in L_h^2(-\infty,0) \quad (\mathrm{Im} - \text{image}).$$

Thus, the relation $v(x,\lambda)f \in L_h^2(-\infty,\infty)$ is possible if λ is a point of degeneracy or singularity of $\Upsilon(\lambda)$, i.e. in view of equalities (2.5), (3.15) and (3.16) the set $\sigma(A_1)\cup\sigma(A_2)$ is of interest for us.

Theorem 3.7. *Let the conditions of corollary 3.1 hold and let the limits* $L_\pm = \lim_{x\to\pm\infty} w_A(x,\lambda)$ *exist* ($\lambda \notin \sigma(A_1)\cup\sigma(A_2)$). *Putting*

$$l_\pm = L_\pm \begin{bmatrix} I_h \\ \mp\mathrm{i}\mu I_h\end{bmatrix} \tag{3.32}$$

suppose additionally that $\det l_\pm \neq 0$. *Then there are no bound states corresponding to* λ, *i.e.* λ *is not an eigenvalue of the Schrödinger equation (2.13).*

Proof. Consider $v(x,\lambda)f$ and put

$$\begin{bmatrix} g_1\\ g_2\end{bmatrix} = \Upsilon(\lambda)^{-1}f \tag{3.33}$$

where $\Upsilon(\lambda)$ ($\lambda\neq 0$) is given by equation (3.16) and $\Upsilon(0) = T_0^{-1}w_A(0,0)$. Let us treat first the case $\lambda \neq 0$. Using equations (3.18) and (3.33) we obtain

$$v(x,\lambda)f = \mathrm{e}^{\mathrm{i}\mu x}(\widetilde{g}_1 + o(1)) + \mathrm{e}^{-\mathrm{i}\mu x}(l_+g_2 + o(1)) \qquad (\widetilde{g}_1 \in \mathbb{C}^h) \qquad \text{when } x\to\infty \tag{3.34}$$

$$v(x,\lambda)f = \mathrm{e}^{\mathrm{i}\mu x}(l_-g_1 + o(1)) + \mathrm{e}^{-\mathrm{i}\mu x}(\widetilde{g}_2 + o(1)) \qquad (\widetilde{g}_2 \in \mathbb{C}^h) \qquad \text{when } x\to-\infty. \tag{3.35}$$

Recall that $\Im\mu \geqslant 0$. Using formulae (3.34) and (3.35) it is clear that the relation $v(x,\lambda)f \in L_h^2(-\infty,\infty)$ yields $l_+g_2 = l_-g_1 = 0$. Taking into account that $\det l_\pm \neq 0$ we obtain $g_1 = g_2 = 0$, i.e. $f = 0$. As v is the fundamental solution, the statement of the theorem is proven for $\lambda\neq 0$. Consider the case $\lambda = 0$ separately. Similar to equation (3.14) we can show that the fundamental solution $v(x,0)$ is given by

$$v(x,0) = [I_h \quad 0]w_A(x,0)\begin{bmatrix} I_h & xI_h\\ 0 & I_h\end{bmatrix} w_A(0,0)^{-1}T_0. \tag{3.36}$$

Using equations (3.33) and (3.36) we now obtain

$$v(x,0)f = x(l_+ + o(1))g_2 + (\widetilde{l} + o(1))g_2 + (l_+ + o(1))g_1 \tag{3.37}$$

($\widetilde{l} = L_+[0 \quad I_h]^*$), when $x\to\infty$. Using equality (3.37) the relation $v(x,\lambda)f \in L_h^2(-\infty,\infty)$ yields $g_1 = g_2 = 0$ again. □

4. State vectors in the $\mathcal{PT}$ symmetric case

In this section we obtain explicit formulae for the $\mathcal{PT}$ symmetric case. For this purpose we assume that $u = 0$, $A = \omega^2$ ($\omega = \omega_1 = \omega_2$), and conditions of proposition 2.4 hold. Recall that u and $\widetilde{u}$ are $m \times m$ matrix functions, ω is an $n \times n$ matrix, and m and n are fixed independent positive integers. Then, taking into account equalities (2.25), we rewrite equalities (3.9) in the form

$$\widetilde{u}(x) = \widetilde{u}(-x)^* = -2X'_{012}(x) \qquad X_{012}(x) = \mathrm{i}\Phi_2(-x)^* S(x)^{-1}\Phi_2(x) \quad (4.1)$$

where according to the first relation in equations (3.6) and (3.7) we have

$$S'(x) = \mathrm{i}\Phi_2(x)\Phi_2(-x)^* \qquad \Phi_2(x) = (\mathrm{e}^{-\mathrm{i}x\omega}\theta_1 + \mathrm{e}^{\mathrm{i}x\omega}\theta_2). \quad (4.2)$$

In view of equalities (2.25), equality (2.5) is rewritten now as

$$w_A(x,\lambda) = I_{2h} - \mathrm{i}J\Pi(-x)^* S(x)^{-1}(\omega^2 - \lambda I_n)^{-1}\Pi(x) \quad (4.3)$$

where

$$\Pi(x) = \Pi_1(x) = [\Phi_1(x) \quad \Phi_2(x)] \qquad \Phi_1(x) = \mathrm{i}\omega(\mathrm{e}^{-\mathrm{i}x\omega}\theta_1 - \mathrm{e}^{\mathrm{i}x\omega}\theta_2). \quad (4.4)$$

From propositions 2.3 and 2.4 and corollary 3.1 follows:

Corollary 4.1.

(i) Let $n \times n$ parameter matrices ω and $S(0) = S(0)^*$ and $n \times h$ parameter matrices θ_1 and θ_2 satisfy the identity

$$\omega^2 S(0) - S(0)(\omega^*)^2 = \mathrm{i}\Pi(0)J\Pi(0)^* \qquad \Pi(0) = [\mathrm{i}\omega(\theta_1 - \theta_2) \quad (\theta_1 + \theta_2)]. \quad (4.5)$$

Then the fundamental solution v of the Schrödinger equation (2.13) with $\widetilde{u}$ defined by equalities (4.1) is given by equalities (3.14) and (3.8), where w_A is constructed in relations (4.2)–(4.4).

(ii) Suppose additionally that $(\theta_1^* \mathrm{e}^{-\mathrm{i}x\omega^*} + \theta_2^* \mathrm{e}^{\mathrm{i}x\omega^*})S(x)^{-1} \in L^2_{h\times n}(-\infty,\infty)$. Then the eigenvectors of ω^2 generate bound states of equation (2.13) with the same eigenvalues, i.e. from $\omega^2 f_a = a f_a$ it follows that

$$\psi_a(x) = \Phi_2(-x)^* S(x)^{-1} f_a = (\theta_1^* \mathrm{e}^{-\mathrm{i}x\omega^*} + \theta_2^* \mathrm{e}^{\mathrm{i}x\omega^*})S(x)^{-1} f_a \quad (4.6)$$

is a bound state of (2.13) with the eigenvalue a.

Thus, the real eigenvalues of ω^2 are of interest and the $\mathcal{PT}$ symmetry is 'spontaneously broken' (there exist unstable bound states) if $\sigma(\omega^2) \not\subset \mathbb{R}$. If $h = 1$, i.e. $\widetilde{u}$ is a scalar function and we consider a scalar Schrödinger equation under the conditions of corollary 4.1, then $\psi_a(-x)^* = f_a^* S(x)^{-1}\Phi_2(x)$ is a bound state with the eigenvalue a^*.

Remark 4.2. If the conditions of (i) of corollary 4.1 are fulfilled and the limit $\varkappa = \lim_{x\to\infty} S(x)^{-1}$ exists, then the scalar product of the eigenfunctions $\psi = \Phi_2(-x)^* S(x)^{-1} f$ and $\widetilde{\psi} = \Phi_2(-x)^* S(x)^{-1}\widetilde{f}$ from the image of $\Phi_2(-x)^* S(x)^{-1}$ is given by

$$(\psi, \widetilde{\psi}) = \widetilde{f}^* K f \qquad K = \mathrm{i}(\varkappa - \varkappa^*). \quad (4.7)$$

Indeed, using equality (2.26) and the first relation in formula (4.2) we have

$$\widetilde{\psi}(-x)^*\psi(x) = \widetilde{f}^* S(x)^{-1}\Phi_2(x)\Phi_2(-x)^* S(x)^{-1} f = \mathrm{i}\widetilde{f}^*(S(x)^{-1})' f. \quad (4.8)$$

Equality (4.7) follows from formulae (2.29) and (4.8).

Example 4.3. Putting $n = 1, \omega = \xi + \mathrm{i}\eta$ ($\xi, \eta \in \mathbb{R}$), and using equalities (4.4), we rewrite identity (4.5) as

$$\xi\eta S(0) = \frac{1}{2}(\mathrm{i}\xi(\theta_1\theta_2^* - \theta_2\theta_1^*) - \eta(\theta_1\theta_1^* - \theta_2\theta_2^*)). \tag{4.9}$$

As $n = 1$, function S is a scalar. From equation (4.2) we obtain

$$S'(x) = \mathrm{i}(\mathrm{e}^{-2\mathrm{i}\xi x}\theta_1\theta_1^* + \mathrm{e}^{2\mathrm{i}\xi x}\theta_2\theta_2^* + \mathrm{e}^{2\eta x}\theta_1\theta_2^* + \mathrm{e}^{-2\eta x}\theta_2\theta_1^*). \tag{4.10}$$

According to equations (4.9) and (4.10) we have

$$S(x) = \frac{\mathrm{i}}{2\eta}(\mathrm{e}^{2\eta x}\theta_1\theta_2^* - \mathrm{e}^{-2\eta x}\theta_2\theta_1^*) - \frac{1}{2\xi}(\mathrm{e}^{-2\mathrm{i}\xi x}\theta_1\theta_1^* - \mathrm{e}^{2\mathrm{i}\xi x}\theta_2\theta_2^*) \qquad \text{if} \quad \xi\eta \neq 0 \tag{4.11}$$

$$S(x) = \frac{\mathrm{i}}{2\eta}(\mathrm{e}^{2\eta x}\theta_1\theta_2^* - \mathrm{e}^{-2\eta x}\theta_2\theta_1^*) + \mathrm{i}x(\theta_1\theta_1^* + \theta_2\theta_2^*) + c \tag{4.12}$$

if $\xi = 0, \eta \neq 0, \theta_1\theta_1^* = \theta_2\theta_2^*$;

$$S(x) = -\frac{1}{2\xi}(\mathrm{e}^{-2\mathrm{i}\xi x}\theta_1\theta_1^* - \mathrm{e}^{2\mathrm{i}\xi x}\theta_2\theta_2^*) + \mathrm{i}x(\theta_1\theta_2^* + \theta_2\theta_1^*) + c \tag{4.13}$$

if $\xi \neq 0, \eta = 0, \theta_1\theta_2^* \in \mathbb{R}$. From (4.1) and relation

$$\Phi_2(-x)^*\Phi_2(x) = (\mathrm{e}^{-2\mathrm{i}\xi x}\theta_1^*\theta_1 + \mathrm{e}^{2\mathrm{i}\xi x}\theta_2^*\theta_2 + \mathrm{e}^{-2\eta x}\theta_1^*\theta_2 + \mathrm{e}^{2\eta x}\theta_2^*\theta_1). \tag{4.14}$$

It follows that

$$\begin{aligned}\widetilde{u}(x) = 4S(x)^{-1}(-\xi(\mathrm{e}^{-2\mathrm{i}\xi x}\theta_1^*\theta_1 - \mathrm{e}^{2\mathrm{i}\xi x}\theta_2^*\theta_2) + \mathrm{i}\eta(\mathrm{e}^{-2\eta x}\theta_1^*\theta_2 - \mathrm{e}^{2\eta x}\theta_2^*\theta_1))\\ + 2\mathrm{i}S(x)^{-2}S'(x)(\mathrm{e}^{-2\mathrm{i}\xi x}\theta_1^*\theta_1 + \mathrm{e}^{2\mathrm{i}\xi x}\theta_2^*\theta_2 + \mathrm{e}^{-2\eta x}\theta_1^*\theta_2 + \mathrm{e}^{2\eta x}\theta_2^*\theta_1).\end{aligned} \tag{4.15}$$

If $\theta_1\theta_2^* \neq 0$ and $\det S(x) \neq 0$, then the conditions of corollary 4.1 are fulfilled in the cases of equations (4.11)–(4.13). Hence the $\mathcal{PT}$ symmetry is spontaneously broken if $\xi\eta \neq 0$.

Remark 4.4. When $\xi\eta = 0, \omega \neq 0, \theta_1\theta_2^* \neq 0$, and $\det S(x) \neq 0$ in the examples (4.12) and (4.13), then the conditions of theorem 3.7 are satisfied. Indeed, using equalities (4.3), (4.4), (4.12) and (4.13) the existence of the limits $L_\pm$ for $\lambda \neq \omega^2$ is immediate. If $\xi = 0, \eta \neq 0$, and $\theta_1\theta_1^* = \theta_2\theta_2^*$ we can put without loss of generality $\eta > 0$ to obtain easily

$$l_+ = I_h - 2\mathrm{i}\eta((\mu + \mathrm{i}\eta)\theta_1\theta_2^*)^{-1}\theta_2^*\theta_1 \qquad l_- = I_h - 2\mathrm{i}\eta((\mu + \mathrm{i}\eta)\theta_2\theta_1^*)^{-1}\theta_1^*\theta_2$$

and

$$l_+^{-1} = I_h + 2\mathrm{i}\eta((\mu - \mathrm{i}\eta)\theta_1\theta_2^*)^{-1}\theta_2^*\theta_1 \qquad l_-^{-1} = I_h + 2\mathrm{i}\eta((\mu - \mathrm{i}\eta)\theta_2\theta_1^*)^{-1}\theta_1^*\theta_2$$

i.e. $l_\pm$ and $l_\pm^{-1}$ are well defined for $\lambda \neq -\eta^2$. In the case $\xi \neq 0, \eta = 0, \theta_1\theta_2^* \in \mathbb{R}$ we have $l_+ = l_- = I_h$ and the conditions of theorem 3.7 are also satisfied. Therefore, the discrete spectrum of the Schrödinger equation with $\mathcal{PT}$ symmetric $\widetilde{u}$ given by equality (4.15) ($\xi\eta = 0$) is the simplest possible; it is concentrated in a real point ω^2. From (ii) of corollary 3.1, the vector function $\psi(x) = (\mathrm{e}^{-\eta x}\theta_1^* + \mathrm{e}^{\eta x}\theta_2^*)S(x)^{-1}$ is a bound state with the real eigenvalue $-\eta^2$ in the example (4.12), and the vector function $\psi(x) = (\mathrm{e}^{-\mathrm{i}\xi x}\theta_1^* + \mathrm{e}^{\mathrm{i}\xi x}\theta_2^*)S(x)^{-1}$ is a bound state with the real eigenvalue ξ^2 in the example (4.13). If $h = 1$ (scalar Schrödinger equation), then $\sqrt{\theta_1\theta_2}\psi$ and $\theta_1\psi$ are $\mathcal{PT}$ symmetric bound states in the examples (4.12) and (4.13), respectively.

When $n = h = 1, \xi = 0, \eta = \frac{1}{2}, \theta_2 = \mathrm{i}\theta_1$, and $c = 0$, we obtain $S(x) = 2|\theta_1|^2(\cosh x + \mathrm{i}x)$. Hence, using equality (4.15) we have

$$\widetilde{u}(x) = 2\left(\left(\frac{\sinh x + \mathrm{i}}{\cosh x + \mathrm{i}x}\right)^2 - \frac{\cosh x}{\cosh x + \mathrm{i}x}\right). \tag{4.16}$$

The bound state ψ has a simple form: $\psi = \frac{e^{-\frac{x}{2}} - i\, e^{\frac{x}{2}}}{\cosh x + i x}$.

Remark 4.5. The potential given by equality (4.16) looks rather close to the potentials $-V_1(\cosh x)^{-2} - iV_2(\cosh x)^{-1} \tanh x$ thoroughly studied in [2, 10, 32], but from the spectral point of view the corresponding Schrödinger equations are essentially different.

5. Conclusion

Thus, the BDT proves a useful tool to study the non-Hermitian and $\mathcal{PT}$ symmetric effects, which provides explicit formulae for the Schrödinger equation with various multiparameter potentials. The simplest cases, from the spectral point of view, are produced and the expression for the sign-indefinite scalar product is simplified, in particular. The reality of the discrete spectrum of the Schrödinger equation proves connected with the reality of the spectrum of the 'generalized' matrix eigenvalue $A = \omega^2$ of the BDT. In this way, some interesting interconnections with the theory of the slowly decaying solutions of the integrable nonlinear equations—positons and harmonic breathers (see [20] and references therein)—are possible.

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